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A Nonparametric Approach to Derivative Asset Pricing

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Abstract

We utilize the method of Bertholon, Monfort and Pegoraro (2006) for pricing European call options based on nonparametric estimation of returns. Densities are estimated using kernel estimation on random samples of a Laplace and mixture of normal distributions. Additionally, an exponential-affine form of the stochastic discount factor allows derivation of closed-form solutions for option prices. Results of the closed form calculations are compared with those calculated using the nonparametric approach. Both are compared to the Black-Scholes formula. Using the Laplace and mixed Gaussian distribution, the nonparametric estimation shows evidence of mispricing by the Black-Scholes model. This technique requires no assumption regarding the form of return distributions.

JEL Classifications: C1, C5, G1.

Keywords: Derivative pricing, European options, nonparametric kernel estimation, Laplace distribution, leptokurtic, mixture of normal distributions, nonparametric pricing, stochastic discount factor.

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1 Introduction

It is well known that the Black-Scholes option pricing model is misspecified in terms of the assumption that the log returns of the underlying asset follow a Gaussian white noise process. Furthermore, empirical studies of return data have demonstrated that stock volatility is not constant over time (Cox, Ingersoll, and Ross, 1985). Additionally, plots of implied volatility versus option strike price give rise to the volatility smile. This volatility smile demonstrates that the volatility of at-the-money options is relatively low, becoming progressively larger for in-the-money and out-of-the-money options. As a result, implied probability distributions calculated using the implied volatility have much fatter tails than the normal distribution. Such a property is termed leptokurticity. Therefore, a higher frequency of lower and higher returns is observed than would be predicted by the Black-Scholes model. Consequently, out-of-the-money options are priced lower if the implied distribution is used in place of the lognormal distribution. This is due to the fact that the option will pay off only if the stock price is above the strike price and the probability of this occurring is lower for the implied probability distribution than for the lognormal distribution. Thus, the implied distribution prices the option lower than the Gaussian distribution of the Black-Scholes model. Therefore, we observe that the Black-Scholes model overprices at-the-money options while under-pricing out-of-the-money and in-the-money call options.

In the literature various methods of addressing these problems have become prominent. One such approach is the generalization of the historical distribution of the underlying stochastic process. Such modifications to the Black-Scholes model include the assumption of time-varying (Merton, 1973) or stochastic volatility (Hull and White, 1987) or even the addition of jump components to the price process (Merton, 1976, and Bates, 1996). Other approaches include alternate derivations of the option pricing formula, pricing using implied distributions and the use of risk-neutral valuation (Cox and Ross, 1976). When pricing options, risk-neutral valuation can be employed directly or, alternatively, an Esscher transform can be used to change the historical distribution into a risk-neutral one. This approach entails using an exponential-affine stochastic discount factor (SDF) to discount the payoff structure of the derivative asset. By pricing the call option using the SDF and assuming absence of arbitrage, it is possible to derive an equation for the price of a European call option based on the risk-neutral distribution.

When the underlying density is not known it is possible to approximate it by estimation. This can be done using two methods: by parametric or nonparametric density estimation. The distinction between the two is that parametric estimation assumes that the type of distribution of the data is known, whereas nonparametric methods estimate the density function directly from the data without any a priori assumptions regarding the underlying distribution.

The density function of the option's underlying asset returns can be estimated parametrically using the Gaussian maximum likelihood estimation (MLE). However, this entails estimating the distribution's mean and variance and such a model may be clearly misspecified. Distributions other than the normal have also been investigated in the literature. Jarrow and Rudd (1989) propose the use of a generalized Edgeworth expansion in order to calculate the joint distribution. This approach takes into account the skewness and the kurtosis of the return density and partially explains the so-called volatility smile. Stable distributions, proposed by Adler, Feldman, and Taqqu (1998), and hyperbolic distributions, are a rich class of probability distributions that take into account fat tails and asymmetry of historical returns. Finite mixture distributions are weighted combinations of a number of probability distributions. Such distributions are useful for modelling historical return data known to contain different groups of observations. Kon (1984) and Tucker and Pond (1998) propose mixtures of distributions where historical stock returns are represented by combinations of normal distributions possessing different variances and possibly different means. Models utilizing a mixture of normal distributions are capable of describing the higher frequencies of stock returns observed near the mean and in the tail areas when compared to normal distribution. Student distributions, proposed by Bollerslev (1987) and Baillie and Bollerslev (1989), are symmetric distributions that can account for the observed fat tails in the distribution of returns.

Parametric approaches have good statistical properties if the true distribution belongs to the family used to build the likelihood function. Gourieroux and Monfort (2006) have developed a method of pricing options based on an exponential-affine stochastic discount factor and an asymmetric Laplace historical distribution (see also Gourieroux and Monfort, 2007). This allows them to obtain a risk-neutral distribution that belongs to the same asymmetric Laplace family. Furthermore, the existence of the risk-neutral distribution allows them to obtain an explicit pricing formula for a European call option. This new formula is considered an extension of the well-known Black-Scholes option pricing equation. It extends the Black-

Scholes model by including parameters to specify the mode and tails of the underlying Laplace distribution. The use of the asymmetric function allows for skewness and leptokurticity in the geometric returns. In their work, Gourieroux and Monfort also extend the approach to obtain a nonparametric pricing formula based on a first-order spline approximation.

In another approach Bertholon, Monfort, and Pegoraro (2006) consider the problem of pricing derivative assets when the stochastic discount factor is again exponential-affine but the geometric return on the underlying asset has dynamics characterized by a mixture of normal distributions. In the paper they consider the static parametric case where the underlying process is a white noise distributed as a mixture of Gaussians. This provides more leptokurticity in the tails of the distribution of returns. Due to the use of an Esscher transform, the subsequent underlying risk-neutral distributions are similar in nature to the historical distributions. Furthermore, the analytic option pricing formula for a mixed normal distribution is found to be a weighted linear combination of Black-Scholes formulas. However, pricing methods based on a parametric family for the historical probability density function (PDF) may exhibit very poor performance if the true distribution does not belong to such a family. This is the reason why Bertholon, Monfort, and Pegoraro (2006) consider a nonparametric approach in which no *a priori* assumption is made on the form of the distribution.

The nonparametric approach can be considered as a particular case of the mixed normal scenario since the nonparametric estimation of a PDF based on a Gaussian kernel is a mixture of normal distributions. In this work we evaluate this nonparametric approach by comparing its results with a closed-form pricing formula for the normal, mixed normal and asymmetric Laplace cases. Moreover, we introduce the Black-Scholes formula to demonstrate that in the case when the underlying distribution is considered to be Gaussian, the nonparametric estimate and exact calculation reproduce the option prices calculated using the Black-Scholes pricing model.

2 Nonparametric Kernel Estimation

Kernel density estimation (KDE) is a nonparametric technique that estimates the underlying probability density of a sample population. In this work we use the KDE method to compute an estimate of the underlying probability distribution function (PDF) of a series of stock returns. This section gives a brief overview of the KDE technique.

2.1 Kernel Density Estimation

We denote the PDF of a random variable X by $f(\cdot)$. If the sample contains n independent observations of X denoted by $x_1, \ldots, x_n$ the kernel density estimator is given by:

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right), \tag{1}$$

where $K(\cdot)$ is the kernel density function and h is termed the bandwidth. In this work we restrict the kernel density function, $K(\cdot)$, to be

$$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right). \tag{2}$$

This is the well known Gaussian kernel density estimator.

2.2 Choice of Kernel Function

A kernel density is estimated using the kernel $K(\cdot)$ with a bandwidth value given by h. In practice, the choice of the bandwidth is much more crucial in comparison to the choice of the kernel function. This is due to the fact that it is possible to rescale the kernel function such that the difference between two given density estimators using two different kernel functions is negligible (Marron and Nolan, 1988).

2.3 Choice of Bandwidth

We now turn our attention to the problem of the appropriate choice of the bandwidth, h. The optimal bandwidth choice is usually computed via minimization of the mean square error (MSE) integral which is given by:

$$MSE\left(\hat{f}_h\right) = \int E\left[\hat{f}_h(x) - f(x)\right]^2 dx, \tag{2a}$$

where $\hat{f}_h(x)$ is the estimated density and $f(x)$ is the true density being estimated. Thus we are minimizing the difference between the true function and its estimate, i.e., the estimation error.

The minimization of the integral leads to an optimal choice of the bandwidth for a given kernel density function $K(\cdot)$. For the Gaussian kernel density estimator an approximate optimal bandwidth choice is the Silverman bandwidth (Silverman, 1981). This value is given by the following equation:

$$h^* = \left(\frac{4\hat{\sigma}^5}{3n}\right)^{\frac{1}{5}} \cong 1.06\hat{\sigma}n^{-\frac{1}{5}}, \tag{2b}$$

where $\hat{\sigma}$ is the sample standard deviation of the random variate samples and n is the number of observations. As the Silverman bandwidth is more of a general rule of thumb, it is not sufficient for use as the best estimate of the bandwidth. Silverman's h^* value is best suited when the density being estimated closely approximates a normal, or Gaussian, PDF. Consequently, it may produce an estimate with insufficient height or be too noisy. Subsequently, in practice, the Silverman bandwidth may need to be adjusted manually in order to effect an acceptable kernel density estimation.

Thus, for a given kernel density function and appropriate choice of bandwidth, we are able to estimate the PDF of an unknown distribution. The method is a powerful and flexible approach for estimating unknown probability density functions. In subsequent sections we will utilize this approach in order to propose a nonparametric approach to option valuation.

3 The Exponential-Affine Stochastic Discount Factor

In discrete time, we assume an incomplete market framework with a risk-free asset and a single risky asset. The risk-free asset between times t and $t+1$ is given by r^f_{t+1} and its value is known at time t. We denote the geometric return on the risky asset having price S_t by $y_{t+1} = \ln\frac{S_{t+1}}{S_t}$.

If an agent makes an investment at time t based on information I_t and we assume the absence of arbitrage, then there exists a SDF $M_{t,t+1}$ that is a function of I_{t+1}. Under such an assumption, the price of an asset with payoff g_{t+1} at time $t+1$ can be written as

$$C_t(g) = E[M_{t,t+1} g_{t+1} | I_t]. \tag{3}$$

Consequently, the price at time t of a European derivative asset with payoff $g(y_{t+1}, \ldots, y_{t+H})$ at $t + H$ is given by

$$\begin{aligned} C_t(g, H) &= E[M_{t,t+1} \ldots M_{t+H-1,t+H}\, g(y_{t+1}, \ldots, y_{t+H}) | I_t] \\ &= E_t[M_{t,t+H}\, g(y_{t+1}, \ldots, y_{t+H})], \end{aligned} \tag{4}$$

where $I_t = (y_t, y_{t-1}, \ldots)$ is the information regarding the current and historical values of the asset price that are available to the investor at time t and $M_{t,t+1}$ is the explicit SDF between dates t and $t+1$. Furthermore, $M_{t,t+1}$ is a function of I_{t+1}.

Since the market is incomplete, there exist multiple SDFs that are compatible with $C_t(g)$ for use in arbitrage pricing. We denote by “exponential-affine” a class of SDFs given by

$$M_{t,t+1} = \exp(\alpha_t y_{t+1} + \beta_t), \tag{5}$$

where α_t and β_t can depend upon the available information set I_t.

Considering the pricing formula at two different dates allows one to obtain two arbitrage free conditions. These conditions enforce a relationship between the SDF and the historical distribution. The absence of arbitrage conditions are:

$$\begin{aligned} E_t\left[M_{t,t+1} \exp\left(r_{t+1}^f\right)\right] &= 1, \\ E_t\left[M_{t,t+1} \exp\left(y_{t+1}\right)\right] &= 1. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} E_t[M_{t,t+1}] &= \exp(-r_{t+1}^f), \\ E_t[M_{t,t+1}] &= \exp(-y_{t+1}). \end{aligned} \tag{7}$$

Because $M_{t,t+1}$ depends on α_t and β_t via equation (5) the solution to the system of equations (6) yields, generally, a unique solution (α_t, β_t) and, consequently, a unique specification of the SDF given by equation (5).

The associated risk-neutral conditional distribution, Q_t of y_{t+1}, is therefore unique. Moreover, given I_t, the risk-neutral distribution has a PDF with respect to the historical distribution given by

$$\frac{M_{t,t+1}}{E_t\left[M_{t,t+1}\right]} = \exp\left(r_{t+1}^f\right)M_{t,t+1}. \tag{8}$$

Thus, an asset providing the payoff $g\left(y_{t+1}\right)$ at time $t+1$ is priced at time t according to

$$\begin{aligned} C_t &= E_t\left[M_{t,t+1}g\left(y_{t+1}\right)\right] \\ &= \exp\left(-r_{t+1}^f\right)E^{Q_t}\left[g\left(y_{t+1}\right)\right]. \end{aligned} \tag{9}$$

This asset pricing equation holds for a time horizon of unity.

4 Option Pricing using an Asymmetric Laplace Distribution

We consider the problem of pricing European call options when the underlying returns are distributed as an asymmetric Laplace distribution. This allows for skewness and leptokurticity and more closely approximates empirically observed return distributions in comparison to the Black-Scholes model assumption of a Gaussian white noise.

The asymmetric Laplace distribution is specified by three parameters: the mode c, and the tail parameters b_0 and b_1. Each tail is described by a different exponential function. Note that c, b_0 and b_1 can be estimated using MLE. The density function is given by Gourieroux and Monfort (2006):

$$f\left(y\right) = \frac{b_0 b_1}{b_0 + b_1}\exp\left[b_0\left(y-c\right)\right] \qquad \text{if} \qquad y < c,$$

$$f\left(y\right) = \frac{b_0 b_1}{b_0 + b_1}\exp\left[-b_1\left(y-c\right)\right] \qquad \text{if} \qquad y > c. \tag{10}$$

This is the probability density function of the asymmetric Laplace distribution. We also specify the following conditions:

$$\begin{aligned} &b_0 > 0, \\ &b_1 > 0, \\ &b_0 + b_1 > 1. \end{aligned} \tag{11}$$

In the above equations b_0 is the parameter that describes the exponential decay of the left tail of the distribution, while b_1 describes the exponential decay rate of the right tail. These conditions guarantee the existence of a unique option price.

As demonstrated in Gourieroux and Monfort (2006), assuming an exponential-affine SDF, the conditional risk-neutral distribution is unique and corresponds to the skewed Laplace distribution $L(b_0, b_1, c)$ with PDF

$$\begin{aligned}\pi(y) &= \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[(b_0+\alpha)(y-c)] \qquad \text{if} \qquad y \le c \\ &= \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b}\exp[-(b_1-\alpha)(y-c)] \qquad \text{if} \qquad y \ge c\,, \end{aligned} \tag{12}$$

where α is the unique solution of the following equation:

$$\exp\left(c - r_{t+1}^{f}\right)(b_0+\alpha)(b_1-\alpha) = (b_0+\alpha+1)(b_1-\alpha-1). \tag{13}$$

Subsequently, the price of the call option written on $\exp(y)$ with payoff $(\exp(y)-k)^{+}$ and a time horizon of unity ($h=1$) is given by Gourieroux and Monfort (2006) and is

$$\begin{aligned} C(k) = C_1(k) &= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp\left[-(b_1-\alpha-1)(\log k - c)\right] \qquad \text{if } \log k \ge c, \\ C(k) = C_2(k) &= 1 - k\exp\left(-r_{t+1}^{f}\right) + \\ &\quad \frac{b_1-\alpha-1}{(b_0+b_1)(b_0+\alpha)}\exp\left[(b_0+\alpha+1)(\log k - c)\right] \qquad \text{if } \log k \le c. \end{aligned} \tag{14}$$

We have two parameters in the risk-neutral density and the call price, c and (b_0+b_1), instead of just one volatility parameter as in the Black-Scholes formula. In this research we take $c = r^f = 0$, r^f being the risk-free rate. This particular case gives a simplified form for the price of the European call option, namely

$$C(k) = C_1(k) = \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp[-(b_1-\alpha-1)(\log k)] \qquad \text{if} \qquad \log k \ge 0\,,$$

$$C(k) = C_2(k) = 1 - k + \frac{b_1-\alpha-1}{(b_0+b_1)(b_0+\alpha)}\exp[(b_0+\alpha+1)(\log k)] \qquad \text{if} \qquad \log k \le 0\,. \tag{15}$$

5 Option Pricing Using a Mixture of Normal Distributions

In this section we introduce the option pricing equations for a mixture of normal distributions. We present the closed-form, or exact, equation followed by the nonparametric pricing equation. A full derivation of the equations has been omitted but is discussed at length in Bertholon, Monfort, and Pegoraro (2006).

5.1 Closed-form (Exact) Pricing Equation

We consider the geometric return y of an underlying asset whose historical distribution is a mixture of J normal distributions. We denote the resulting PDF by $MN(n, w_j, \mu_j, \sigma_j^2)$ where μ_j, σ_j^2 and w_j are the mean, variance, and weight, respectively, of distribution *j*. The probability distribution function of a mixture of normal distributions is given by

$$f(y) = \sum_{j=1}^{J} w_j n(y; \mu_j, \sigma_j^2), \tag{16}$$

where $n(y; \mu_j, \sigma_j^2)$ is a normal distribution, given by

$$n(y; \mu_j, \sigma_j^2) = \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left(\frac{-(y - \mu_j)^2}{2\sigma_j^2} \right) \quad \text{for} \quad j = 1, \ldots, J. \tag{17}$$

For example, assuming a mixture of $J = 2$ Gaussian distributions, we have five parameters: two means (μ_1, μ_2), two variances (σ_1^2, σ_2^2) and the weights w_j of a particular distribution in the mixture. By choosing the appropriate set of parameters it is possible to alter the shape of the PDF.

For normalization purposes, the weights assigned to the components of the distribution must satisfy

$$\begin{gathered} 0 \le w_j \le 1, \\ \sum_{j=1}^{J} w_j = 1. \end{gathered} \tag{18}$$

Assuming that the SDF is given by (see Gourieroux and Monfort, 2007)

$$M_{t,t+1} = \exp(\alpha y_{t+1} + \beta), \tag{19}$$

it can be shown (see Bertholon, Monfort, and Pegoraro, 2006) that there exists a unique value of α given by

$$\sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right)\left[\exp\left(\mu_j + \sigma_j^2\alpha + \frac{\sigma_j^2}{2}\right) - \exp\left(r^f\right)\right] = 0, \tag{20}$$

where p_j is a weighting factor. Bertholon, Monfort, and Pegoraro (2006) also show that the form of the risk-neutral distribution for a mixture of normal distributions is

$$f^Q(y) = \sum_{j=1}^{J} \nu_j n\left(y; \mu_j + \alpha\sigma_j^2, \sigma_j^2\right), \tag{21}$$

where the weighting factor ν_j is denoted by

$$\nu_j = \frac{p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right)}{\sum_{j=1}^{n} p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right)}, \tag{22}$$

and satisfies the following two conditions:

$$\begin{aligned} &0 \le \nu_j \le 1, \\ &\sum_{j=1}^{J} \nu_j = 1. \end{aligned} \tag{23}$$

It is important to note that, unlike the Black-Scholes model, the risk-neutral distribution for a mixture of normals does not only depend on the variance. Thus, it accounts not only for the volatility, but also the return.

Based on the exponential-affine SDF model, Bertholon, Monfort, and Pegoraro (2006) give the price of a European call option having payoff $\exp(y_{t+1} - \kappa)^+$ and maturity of one period as

$$\begin{aligned} C(\kappa) &= \exp\left(-r^f\right)E^Q[\exp y - \kappa]^+ \\ &= \sum_{j=1}^{J} \nu_j \gamma_j C_{BS}\left(\sigma_j^2, \frac{\kappa}{\gamma}\right), \end{aligned} \tag{24}$$

where

$$\gamma_j = \exp\left(\mu_j + \alpha\sigma_j^2 - r^f \frac{\sigma_j^2}{2}\right), \tag{25}$$

and $C_{BS}(\cdot)$ is the one period Black-Scholes formula with volatility σ_j^2 and moneyness strike κ / γ_j. Here κ is the relative strike price $(\kappa = K / S_t)$ and S_t (the price of the underlying asset) is normalized to unity. For convenience, we take r^f (the risk-free rate) to be zero.

This formula is a generalization of the Black-Scholes formula. It is a linear combination of J Black-Scholes equations and depends not only on the variances but also on the means of the normal distributions in the mixture. If the index $j = 1$, we have the standard Black-Scholes formula in the normal case. For this reason, we can call it the *modified Black-Scholes formula*.

It can also be shown that the product of γ_j and ν_j satisfies the following conditions:

$$\begin{gathered} 0 \le \nu_j \gamma_j \le 1, \\ \sum_{j=1}^{J} \nu_j \gamma_j = 1. \end{gathered} \tag{26}$$

Although these results consider the special case of a single-period geometric return, it is possible to extend the considerations above to a time horizon of $h > 1$. Because this complicates some of the pricing procedures we do not present these results in this particular work.

5.2 Nonparametric Pricing Equation

We consider now the nonparametric valuation method. In this case we use a Gaussian kernel density estimator to approximate the probability distribution function of the underlying asset returns. We assume the geometric returns are independently and identically distributed (IID) and that their distribution is unknown.

We price the option by examining a series of observations of the geometric return given by $y_1, y_2, \ldots, y_n$. From equation (1) the Gaussian kernel density estimate can be written as

$$f(y) = \sum_{j=1}^{n} \frac{1}{n} n\left(y; y_j, h^2\right). \tag{27}$$

Conveniently, this estimated PDF is a mixture of normal distributions. Thus, in this case, we can apply the pricing method described in the previous section. The equation yielding the unique value of α parallels equation (20) and is given by

$$\sum_{j=1}^{n} \exp\left(\alpha y_j\right)\left[\exp\left(y_j + h^2\alpha + \frac{h^2}{2}\right) - \exp r^f\right] = 0 . \tag{28}$$

Once again, we require the risk-neutral probability density. This is derived by Bertholon, Monfort, and Pegoraro (2006):

$$f^Q(y) = \sum_{j=1}^{n} \nu_j n\left(y; y_j + \alpha h^2, h^2\right), \tag{29}$$

where, for $j = 1, \ldots, n$, the value of ν_j is

$$\nu_j = \frac{\exp\left(\alpha y_j\right)}{\sum_{j=1}^{n} \exp\left(\alpha y_j\right)}, \tag{30}$$

and is consistent with the definition of a weighting factor.

Because we have assumed a Gaussian kernel function, we arrive at the price of a plain vanilla European call option that consists of a linear combination of n Black-Scholes formulas (see Bertholon, Monfort, and Pegoraro, 2006):

$$C(\kappa) = \sum_{j=1}^{n} \nu'_j C_{BS}\left(h^2, \frac{\kappa}{\gamma_j}\right), \tag{31}$$

where

$$\nu'_j = \frac{\exp\left((\alpha+1)y_j\right)}{\sum_{j=1}^{n} \exp\left((\alpha+1)y_j\right)} \quad \text{and} \quad \gamma_j = \exp\left(y_j + \alpha h^2 - r + \frac{h^2}{2}\right).$$

Thus, by using the kernel density estimation approach, it is possible to derive an explicit closed-form option valuation formula for any kind of distribution. In addition, we have also obtained an analytic solution for the risk-neutral distribution.

6 Results

6.1 Asymmetric Laplace Distribution

The asymmetric Laplace distribution is characterized by three parameters. These include decay constants for the left and right tails of the PDF and the mode which determines the position of the peak of the distribution. Having simulated stock returns according to the distributions specified previously, we used the methodology discussed to price European call options written on such distributions. Repeating the same procedure utilized in the previous cases, we priced a European call option for various values of moneyness strike. The results of the calculations are presented in Figures 1-3.

Figure 1

Option price vs. moneyness strike for the asymmetric Laplace distribution.

(Parameters: b_0=1.1, b_1=1.4, and c=0.0. Bandwidth: h=0.245.)

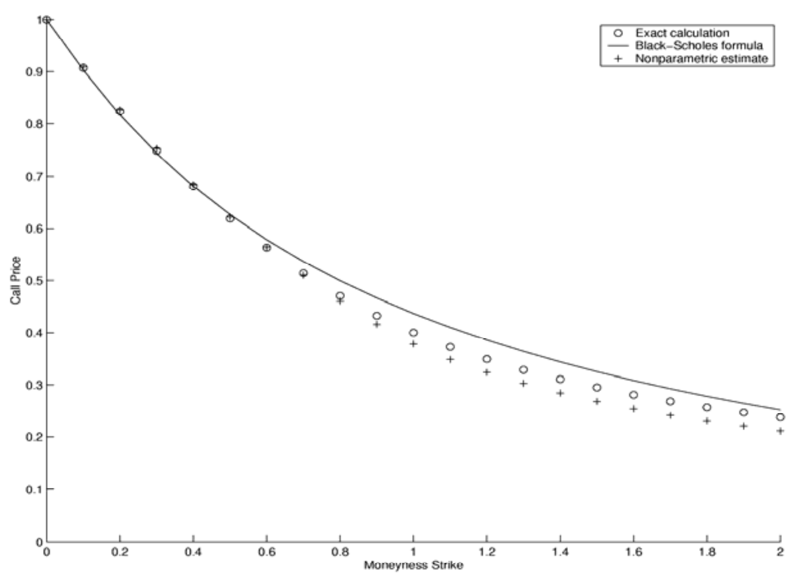


Figure 2

Option price vs. moneyness strike for the asymmetric Laplace distribution.

(Parameters: b_0=1.2, b_1=1.6, and c=0.0. Bandwidth: h=0.357.)

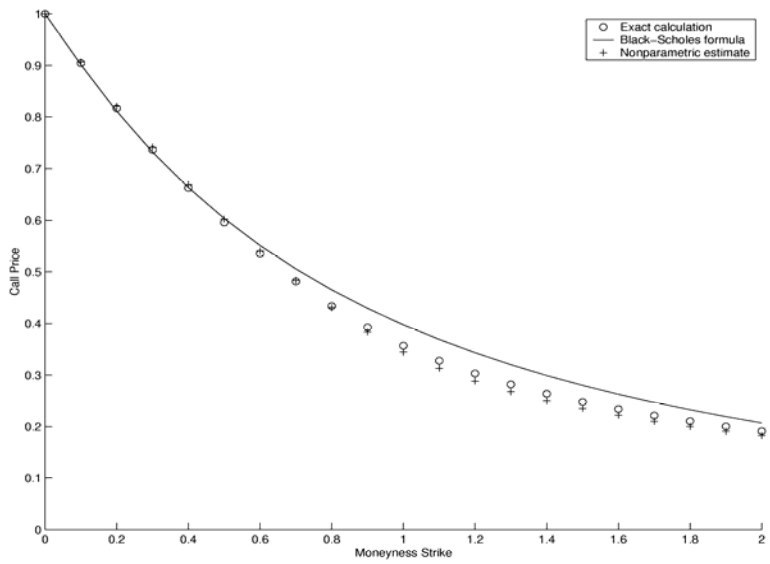
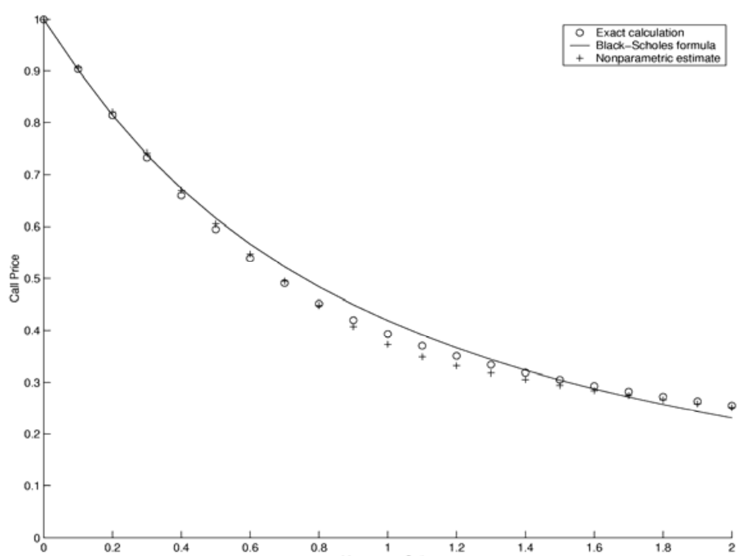


Figure 3

Option price vs. moneyness strike for the asymmetric Laplace distribution.

(Parameters: b_0=1.1, b_1=1.6, and c=-0.30. Bandwidth: h=0.275.)



Examining Figure 1 for the asymmetric Laplace option price calculation, the effect of the increased leptokurticity (fat tails) and skewness is clearly evident. Immediately one can see that the option price calculated using the asymmetric Laplace distribution is significantly lower for at-the-money to out-of-the-money options than that calculated using the Black-Scholes formula. Furthermore, we can see from the graph slight evidence that the Black-Scholes model under-prices in-the-money options for the associated choice of parameters. This behaviour stems from the fact that the Gaussian distribution is considered to be mesokurtic, i.e., possessing a kurtosis of zero. This is the main reason behind the higher option prices observed as we price call options using distributions with increasingly large kurtosis. Agreement between the exact and nonparametric calculations begins to diverge with increasing moneyness.

Figure 2 shows the price of the call option plotted against the moneyness strike. In this particular set of parameters, the right tail coefficient is chosen to be larger than the left. We can see that the Black-Scholes formula overprices out-of-the-money options at-the-money options. Note that there is slight indication of under-pricing by the Black-Scholes model for deeply out-of-the-money options. Again we observe the effects of the increased leptokurticity. Note that in this case, the nonparametric estimate of the call option price is slightly less than both the Black-Scholes and exact option prices for at-the-money options evidencing the sensitivity to bandwidth.

Finally, we examine Figure 3. In this case the mode of the distribution has been chosen to be different from zero. This introduces further skewness. Figure 3 shows that, while the Black-Scholes formula overprices at-the-money and weakly out-of-the-money options, it under-prices deeply out-of-the-money options. Similar behaviour has been noted by Bertholon, Monfort, and Pegoraro (2006). Because of the greater skewness and kurtosis of the Laplace distribution, compared to the mixture of normal distributions, the effects of these increases are more pronounced. Agreement between the nonparametric and exact calculations is clearly evident.

The Laplace distribution with appropriately chosen parameters and bandwidth seems to reproduce the stylized facts of empirically observed stock return distributions. While this distribution seems to accurately model these observations, it is highly sensitive to the choice of bandwidth. As such, future research may include development of an appropriate bandwidth choice criterion.

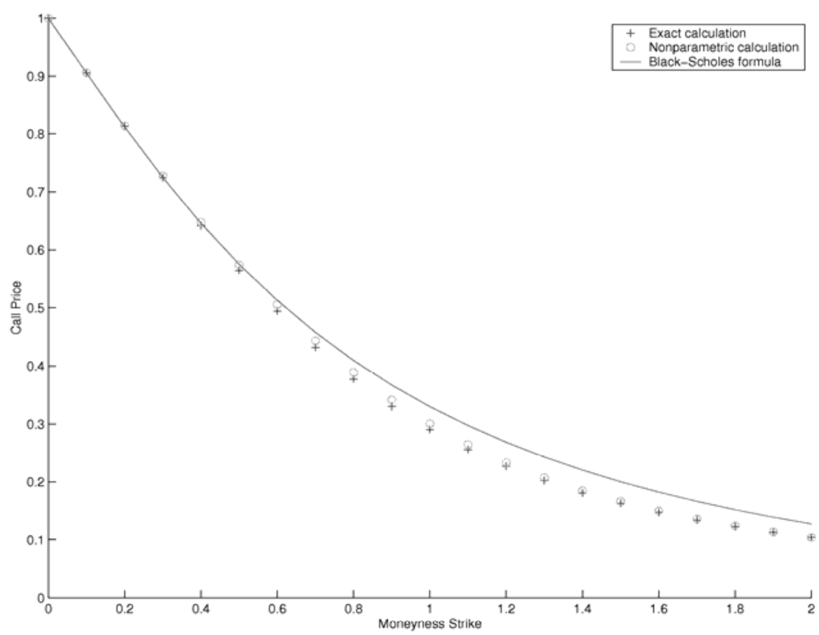
6.2 Mixed Normal Distribution

We constructed a mixture of distributions by combining two individual Gaussian PDFs each having different parameters. The total number of parameters required to describe the mixture is five. These include the two individual means and two individual variances. The fifth, and last, parameter is a weighting factor assigned to one particular distribution such that the total weight for both distributions sums to unity. Figures 4-7 show the price of the European call option calculated using the nonparametric and exact mixed normal approach as well as the option price calculated via evaluation of the Black-Scholes option pricing formula.

Figure 4

Option price vs. moneyness strike.

(Parameters: $\mu_1 = 0.0,\ \mu_2 = -0.5,\ \sigma_1^2 = 0.05,\ \sigma_2^2 = 0.95,\ w_1 = 0.55,\ w_2 = 0.45,\ h = 0.185.$)



In Figure 4 we see that the Black-Scholes model overprices weakly in-the-money options. Evidence of overpricing is strong for at-the-money and out-of-the-money options for this particular choice of parameters. This can be seen clearly in the range of moneyness strikes between [0.8, 2.0]. We note that the difference between the two means introduces skewness into the simulated data. Thus, option prices calculated using the exact and nonparametric method are less than those predicted by the Black-Scholes model.

In Figure 5 we see results similar to the previous case. The effect of the difference in variance is evident in the moneyness strike range of [0.6, 2]. It can be seen that deeply out-of-the-money options are overpriced by the Black-Scholes model. We see good agreement between the nonparametric and exact calculations with some discrepancy in the at-the-money region. We can infer that this is perhaps caused by the number of simulations used in the calculation and, additionally, to the choice of bandwidth.

Figure 5

Option price vs. moneyness strike.

(Parameters: $\mu_1 = 0.0,\ \mu_2 = -0.5,\ \sigma_1^2 = 0.05,\ \sigma_2^2 = 0.65,\ w_1 = 0.55,\ w_2 = 0.45,\ h = 0.165.$)

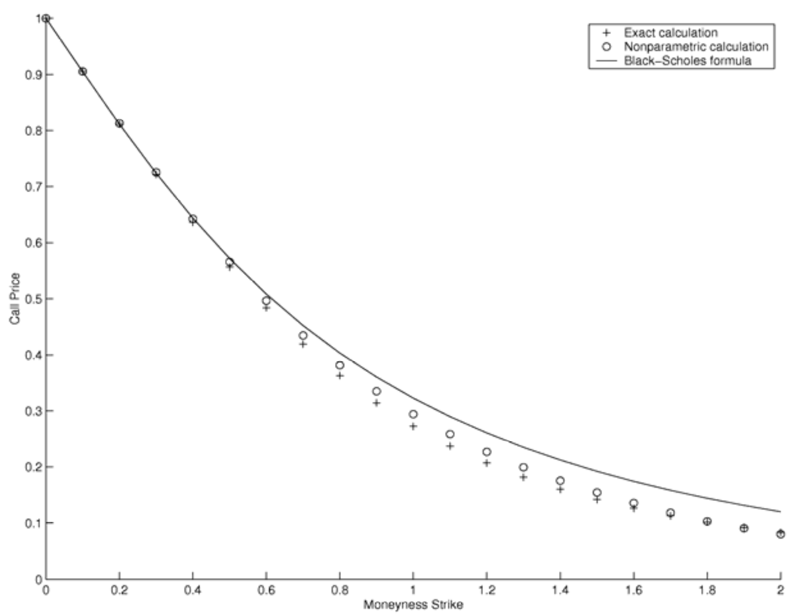
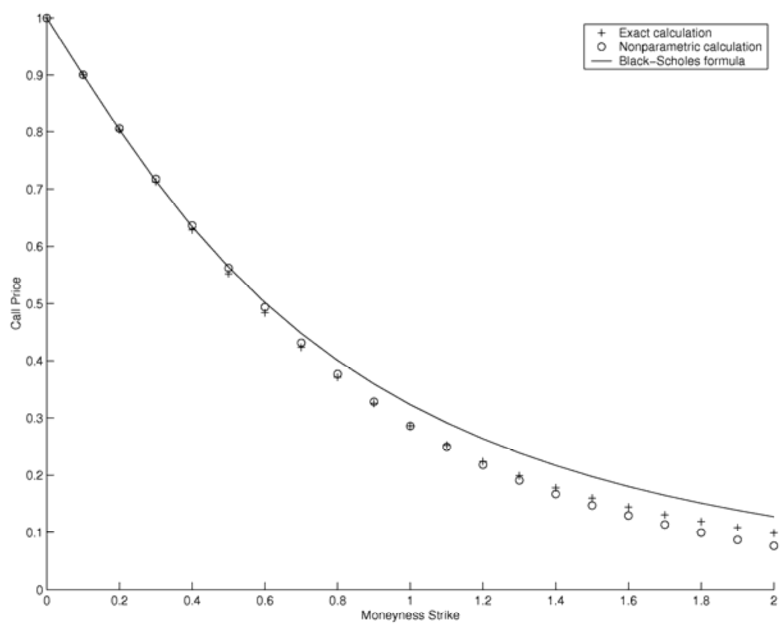


Figure 6 shows a pronounced difference between the exact, nonparametric and Black-Scholes option price calculations in the deeply out-of-the-money region. However, the exact and nonparametric calculations exhibit more agreement between themselves than the Black-Scholes prices. Looking closely, it can be seen that the Black-Scholes model excessively overprices at-the-money options, as well as out-of-the-money to deeply out-of-the-money options. This may be attributed to the presence of skewness in the estimated underlying distribution. We note that the difference between the Black-Scholes and the nonparametric approach is increasing with the skewness. Furthermore, the presence of skewness more accurately resembles observed stock returns.

Figure 6

Option price vs. moneyness strike.

(Parameters: $\mu_1 = 0.0,\ \mu_2 = -0.5,\ \sigma_1^2 = 0.05,\ \sigma_2^2 = 0.65$, $w_1 = 0.35,\ w_2 = 0.65,\ h = 0.155$.)

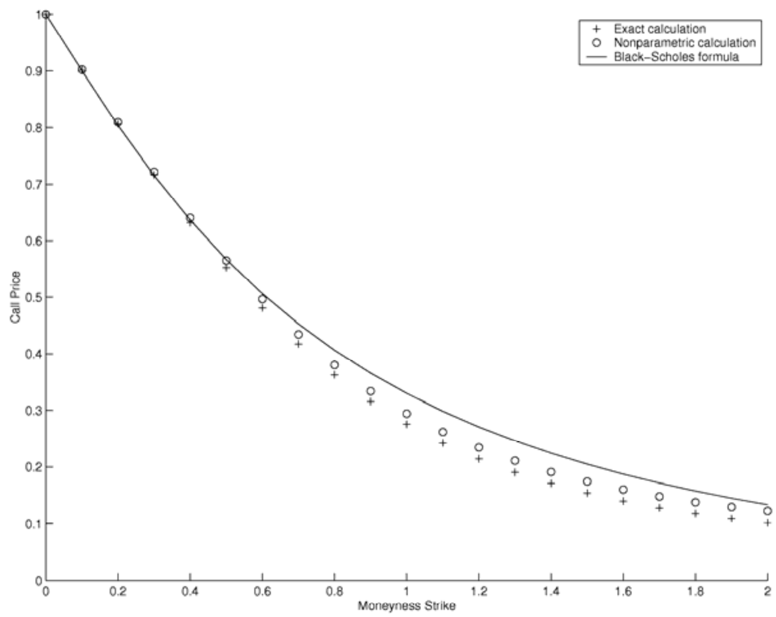


Finally, Figure 7 illustrates the result of the final pricing calculation. It can be clearly seen that the Black-Scholes model overprices at-the-money options. Note that for deeply out-of-the-money options, the nonparametric and exact curves closely match the Black-Scholes curve. For very deeply out-of-the-money options, the Black-Scholes model may underprice the option. Such behaviour of the Black-Scholes model has been empirically observed using actual stock returns. The choice of the different means introduces skewness to the distribution, while the different variances increase the tail weights or leptokurticity. Furthermore, we can see that the nonparametric calculation of the option price is in close agreement to the exact or “true” option price.

Figure 7

Option price vs. moneyness strike.

(Parameters: $\mu_1 = 0.0,\ \mu_2 = -0.4,\ \sigma_1^2 = 0.05,\ \sigma_2^2 = 1.95,\ w_1 = 0.7,\ w_2 = 0.3,\ h = 0.225.$)



7 Effect of Number of Simulations

In this section we use the mixture of normal distributions to study the effect of increasing the number of sample points. We increased the sample points from 100 to 500 in increments of one hundred samples and observed the effect on the call option price curve. To clarify the effect, we have considered at-the-money options restricting the moneyness strike values to the interval $[0.6,\ 1.4]$.

Due to the random generation of the samples, we have illustrated the effect for two separate trials. Figures 8 and 9 show the effect on the call option price of increasing the number of simulations from 100 to 500 samples. The calculations were performed for a mixture of normal distributions.

Figure 8 clearly shows the effect of increasing the number of samples to 500. Note that as the number of samples approaches 500 agreement between the nonparametric and exact calculation improves considerably. In Figure 9 we observe a similar trend as in Figure 8.

Increasing the number of samples gives greater convergence to the exact calculation. As the number of samples is increased, the statistical moments of the estimated distribution approach those of the exact distribution, thereby effecting greater agreement between the various calculations of the option price.

Figure 8
Trial 1: the effect on the call option price of increasing the number of simulations from 100 to 500 samples.

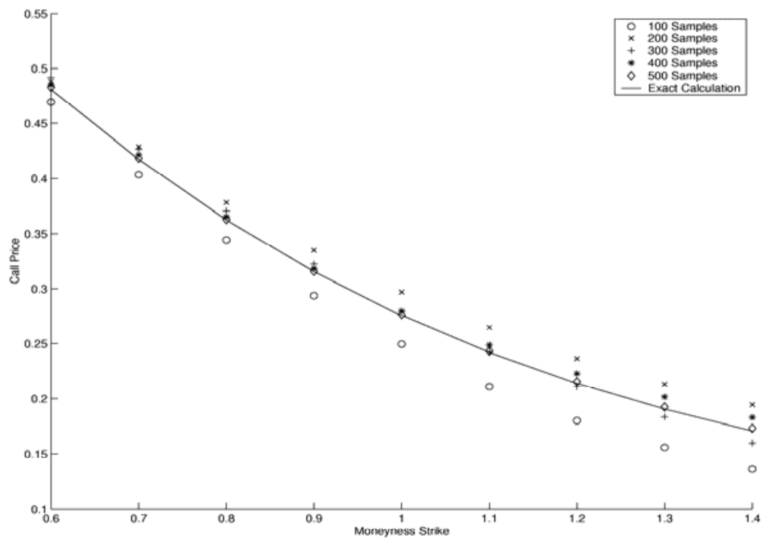
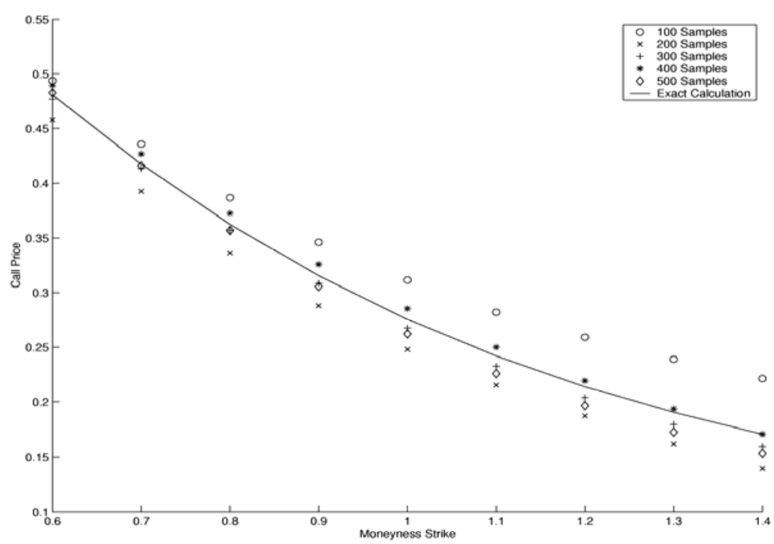


Figure 9
Trial 2: the effect on the call option price of increasing the number of simulations from 100 to 500 samples.



8 Conclusion

In this paper we presented a nonparametric KDE technique as a tool for pricing options or derivatives in general. The nonparametric approach avoids the need to make assumptions regarding the true form of the underlying PDF. The literature on option pricing is abundant but there are not many publications on option pricing using nonparametric kernel estimation. Our contribution has been to add an empirical study on this particular topic.

Derivative prices depend on the estimation of a probability density function from observations. KDE is an efficient technique to estimate the densities in comparison with classical parametric methods in cases where the underlying distribution is never known, or when it is hard to derive an analytical option pricing equation. Moreover, a nonparametric approach to pricing can generate prices that are robust to parameter specification errors. However, the kernel estimation technique requires a large number of calculations.

The use of nonparametric pricing techniques also allows for pricing European call options using leptokurtic distributions such as the mixed normal and asymmetric Laplacian PDFs. The increased kurtosis of such distributions, compared to the Gaussian distribution, leads to option prices lower than those predicted by the Black-Scholes model, particularly for at-the-money to out-of-the-money options. Such behaviour may have a noticeable effect on the implied volatility smiles predicted by the Black-Scholes model.

To price options using the nonparametric method, random samples from the Gaussian, mixed Gaussian and asymmetric Laplace distributions were simulated and the nonparametric and exact formulas were used to price various European call options. Our results indicate that the distributions with the greatest kurtosis lead to out-of-the-money option prices slightly lower than those predicted by the Black-Scholes model, while highlighting the overpricing of at-the-money options by the Black-Scholes model. This is a direct result of the increased weight in the tails of the respective PDFs. Furthermore, implementation of the pricing formulas was found to be simple and expedient even given the large number of calculations required in the exact formula pricing case. This is a direct result of the evaluation of the closed form solutions of the pricing formulae as given in Bertholon, Monfort, and Pegoraro (2006), and Gourieroux and Monfort (2006).

From the results presented, we also observed that the nonparametric estimation of the underlying distributions was very much in line with the theoretically predicted distribution shapes. This is evidence that the method is an efficient and accurate simulation procedure. Furthermore, we note that the option prices calculated using the nonparametric technique are in close agreement to the option prices calculated using the closed-form exact solutions provided in this work.

One further important observation is that the nonparametric method is in excellent agreement with the Black-Scholes model when the distribution is Gaussian, even if we make no *a priori* assumption that the distribution is normal. In light of these results, we can conclude that the nonparametric method is valid regardless of the underlying distribution.

One drawback of the method is the use of a bandwidth in the nonparametric estimation procedure. Although the Silverman bandwidth is acceptable as an initial estimate, it was found that further improvement and fit accuracy could be obtained if the bandwidth was adjusted manually. This confirms that, in terms of pricing options, the choice of the bandwidth is more significant than the choice of the kernel function. This finding certainly warrants further investigation.

Another worthwhile pursuit would be an investigation into how to properly choose parameters for the parametric estimates discussed in this work. Such an approach could utilize a maximum likelihood estimator to estimate the parametric PDFs.

In this research we priced options using a time horizon of unity. Although calculations for any horizon greater than unity become more complex, particularly in the case of the Laplace distribution, further investigation and research in this area is needed.

Finally, one strong assumption made in this paper was that returns are IID. A next step would be to consider returns that are not IID. Other possible future extensions may include investigation of the effect of various distributions on the volatility smile or to study the behaviour of the implied volatility.

In light of our investigation we can conclude that the nonparametric option pricing technique is an easily implemented, accurate and practical method for pricing European call options written on leptokurtic distributions. The procedure is attractive because it avoids the need to make particular assumptions regarding the PDF of the asset returns. However, the method requires some refinement, most notably, in choosing the bandwidth and for pricing options with arbitrary maturities.

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